Ideals of Graph Homomorphisms

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Ideals of Graph Homomorphisms

• Joint work with Patrik Norén at KTH.
• Funded by the Miller Institute for Basic Research in Science at UC Berkeley.
• This presentation is on my homepage math.berkeley.edu/~alex/
• Based on the preprint “Ideals of Graph Homomorphisms”, arXiv:1002.4679
• Scripts for 42 on math.kth.se/~pnore/
Motivation

• The ideals of graph homomorphisms are natural generalizations of toric ideals used in algebraic statistics.
• The polytopes defining the toric ideals of graph homomorphisms are important in optimization theory.
• Graphs and their homomorphisms is a category. Instead of just creating toric ideals from graphs, all of the category should be used.
• The ideals of graph homomorphisms could be studied in their on right. They have an amazing amount of beautiful properties.

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Ideals from graphs in algebraic statistics

• A statistical model is defined from a graph $G$ and random variables for each vertex of $G$.

• A toric ideal $I(G)$ is defined from the model.

• If the graph or the type of random variables is changed, how is that reflected in $I(G)$?

• If $G_1$ is a subgraph of $G_2$, or there is a map from $G_1$ to $G_2$, how are $I(G_1)$ and $I(G_2)$ related?
Homomorphisms

• Let $\text{Hom}(A,B)$ be the set of maps between the graphs/rings/spaces $A$ and $B$.

• Any map from $A$ to $B$ induces
  – a map from $\text{Hom}(C,A)$ to $\text{Hom}(C,B)$, and
  – a map from $\text{Hom}(B,C)$ to $\text{Hom}(A,C)$.

• If we construct “natural” ideals $I_{\text{Hom}(,)}$ on $\text{Hom}(,)$ then any map from $A$ to $B$ induces
  – a map from $I_{\text{Hom}(C,A)}$ to $I_{\text{Hom}(C,B)}$, and
  – a map from $I_{\text{Hom}(B,C)}$ to $I_{\text{Hom}(A,C)}$.

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Graph homomorphisms

• A graph homomorphism from $G$ to $H$ is a map $f$ from $V(G)$ to $V(H)$ such that if $uv$ is an edge of $G$ then $f(u)f(v)$ is an edge of $H$.

• Examples of $\text{Hom}(G,H)$
  – $H$ is a complete graph on $[n]$ with all loops: Arbitrary assignment of $1,2,...,n$ to the vertices.
  – $H$ is a complete graph on $[n]$ without loops: $n$-colorings of $G$.
  – $H$ is an edge with one loop: The independent sets of $G$.

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Ideals of Graph Homomorphisms

• Let $G$ and $H$ be graphs. Define polynomial rings:
  • $R_{G \to H} = k[ r_\phi : \phi$ is a graph homomorphism from $G$ to $H ].$
  • $S_{G \to H} = k[ s_\phi : e$ is an edge of $G$, and $\phi$ is a graph homomorphism from $e$ to $H ].$

• Define a ring homomorphism $\Phi_{G \to H}$ from $R_{G \to H}$ to $S_{G \to H}$ by sending $r_\phi$ to the product of $s_{\phi|e}$ for all edges $e$ of $G$.

• Definition: The ideal of graph homomorphisms from $G$ to $H$, $I_{G \to H}$, is the kernel of $\Phi_{G \to H}$.

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The ideal of the 3-path $\rightarrow$ 2-path

<table>
<thead>
<tr>
<th>Map</th>
<th>Variable</th>
<th>Image of $\Phi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1234→1232</td>
<td>$r_{1234\rightarrow1232}$</td>
<td>$s_{12}\rightarrow12s_{23}\rightarrow23s_{34}\rightarrow32$</td>
</tr>
<tr>
<td>1234→1212</td>
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<td>$s_{12}\rightarrow12s_{23}\rightarrow21s_{34}\rightarrow12$</td>
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<tr>
<td>1234→2121</td>
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<tr>
<td>1234→2123</td>
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<tr>
<td>1234→2321</td>
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<td>1234→2323</td>
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<tr>
<td>1234→3212</td>
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<tr>
<td>1234→3232</td>
<td>$r_{1234\rightarrow3232}$</td>
<td>$s_{12}\rightarrow32s_{23}\rightarrow23s_{34}\rightarrow32$</td>
</tr>
</tbody>
</table>

Markov basis

$<r_{1234\rightarrow1212}, r_{1234\rightarrow3232}$
$- r_{1234\rightarrow1232}, r_{1234\rightarrow3212}, r_{1234\rightarrow2121}, r_{1234\rightarrow2323},$ $r_{1234\rightarrow2123}, r_{1234\rightarrow2321} >$

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Examples of ideals of graph homomorphisms $I_{G \rightarrow H}$

- **H** is a complete graph on $[n]$ with all loops: Arbitrary assignment of $1, 2, ..., n$ to the vertices. Studied in algebraic statistics.

- **H** is a complete graph on $[n]$ without loops: $n$-colorings of $G$. Obstructions of graph colorings.

- **H** is an edge with one loop: The independent sets of $G$. Defined by the stable set polytope. Hibi’s ASL rings is a special case.

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Examples of some theorems

• Maps between graphs induce maps between ideals (and quotients). Reflects on the Markov width.

• There is a quadratic square-free Gröbner base of $I_{G \to H}$ if
  – the graph $G$ is a forest, or
  – the graph $H$ is an edge with one loop, and $G$ is (almost) bipartite.

• For $H$ an edge with one loop, any Markov width of $I_{G \to H}$ is possible.
Read more about it here

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