Ideals of Graph Homomorphisms

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Ideals of Graph Homomorphisms

- Joint work with Patrik Norén at KTH.
- Funded by the Miller Institute for Basic Research in Science at UC Berkeley.
- This presentation is on my homepage math.berkeley.edu/~alex/
- Based on the preprint “Ideals of Graph Homomorphisms”, arXiv:1002.4679
- Scripts for 42 on math.kth.se/~pnore/

Motivation

- The ideals of graph homomorphisms are natural generalizations of toric ideals used in algebraic statistics.
- The polytopes defining the toric ideals of graph homomorphisms are important in optimization theory.
- Graphs and their homomorphisms is a category. Instead of just creating toric ideals from graphs, all of the category should be used.
- The ideals of graph homomorphisms could be studied in their on right. They have an amazing amount of beautiful properties.

Ideals from graphs in algebraic statistics

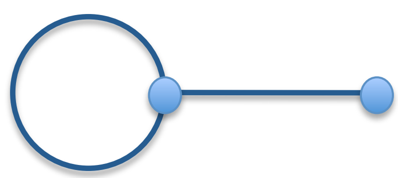
- A statistical model is defined from a graph **G** and random variables for each vertex of **G**.
- A toric ideal **I(G)** is defined from the model.
- If the graph or the type of random variables is changed, how is that reflected in **I(G)**?
- If $\mathbf{G_1}$ is a subgraph of $\mathbf{G_2}$, or there is a map from $\mathbf{G_1}$ to $\mathbf{G_2}$, how are $\mathbf{I(G_1)}$ and $\mathbf{I(G_2)}$ related?

Homomorphisms

- Let **Hom(A,B)** be the set of maps between the graphs/rings/spaces **A** and **B**.
- Any map from **A** to **B** induces
  - a map from **Hom(C,A)** to **Hom(C,B)**, and
  - a map from **Hom(B,C)** to **Hom(A,C)**.
- If we construct "natural" ideals $\mathbf{I}_{\mathbf{Hom(,)}}$ on **Hom(,)** then any map from **A** to **B** induces
  - a map from $\mathbf{I}_{\mathbf{Hom(C,A)}}$ to $\mathbf{I}_{\mathbf{Hom(C,B)}}$, and
  - a map from $\mathbf{I}_{\mathbf{Hom(B,C)}}$ to $\mathbf{I}_{\mathbf{Hom(A,C)}}$.

Graph homomorphisms

- A graph homomorphism from **G** to **H** is a map **f** from **V(G)** to **V(H)** such that if **uv** is an edge of **G** then **f(u)f(v)** is an edge of **H**.
- Examples of **Hom(G,H)**
  - **H** is a complete graph on **[n]** with all loops: Arbitrary assignment of **1,2,…,n** to the vertices.
  - **H** is a complete graph on **[n]** without loops: **n**-colorings of **G**.
  - **H** is an edge with one loop: The independent sets of **G**.

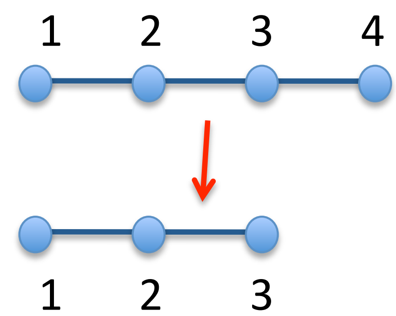


Ideals of Graph Homomorphisms

- Let **G** and **H** be graphs. Define polynomial rings:
- $\mathbf{R_{G \to H}} = \mathbf{k[\ r_\phi :}$ ϕ is a graph homomorphism from **G** to **H]**.
- $\mathbf{S_{G \to H}} = \mathbf{k[\ s_\phi :}$ **e** is an edge of **G**, and ϕ is a graph homomorphism from **e** to **H]**.
- Define a ring homomorphism $\Phi_{\mathbf{G \to H}}$ from $\mathbf{R_{G \to H}}$ to $\mathbf{S_{G \to H}}$ by sending $\mathbf{r_\phi}$ to the product of $\mathbf{s_{\phi|e}}$ for all edges **e** of **G**.
- Definition: The *ideal of graph homomorphisms from* **G** *to* **H**, $\mathbf{I_{G \to H}}$, is the kernel of $\Phi_{\mathbf{G \to H}}$.

The ideal of the 3-path → 2-path

Map	Variable	Image of Φ
1234→1232	$r_{1234\to1232}$	$s_{12\to12}s_{23\to23}s_{34\to32}$
1234→1212	$r_{1234\to1212}$	$s_{12\to12}s_{23\to21}s_{34\to12}$
1234→2121	$r_{1234\to2121}$	$s_{12\to21}s_{23\to12}s_{34\to21}$
1234→2123	$r_{1234\to2123}$	$s_{12\to21}s_{23\to12}s_{34\to23}$
1234→2321	$r_{1234\to2321}$	$s_{12\to23}s_{23\to32}s_{34\to21}$
1234→2323	$r_{1234\to2323}$	$s_{12\to23}s_{23\to32}s_{34\to23}$
1234→3212	$r_{1234\to3212}$	$s_{12\to32}s_{23\to21}s_{34\to12}$
1234→3232	$r_{1234\to3232}$	$s_{12\to32}s_{23\to23}s_{34\to32}$

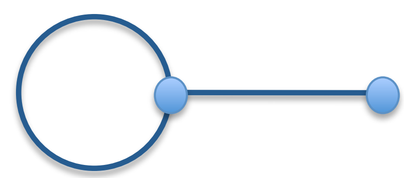


Markov basis

$$\langle r_{1234\to1212}r_{1234\to3232} - r_{1234\to1232}r_{1234\to3212}, \; r_{1234\to2121}r_{1234\to2323} - r_{1234\to2123}r_{1234\to2321}\rangle$$

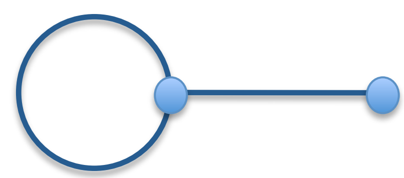
Examples of ideals of graph homomorphisms $\mathbf{I_{G \to H}}$

- **H** is a complete graph on **[n]** with all loops: Arbitrary assignment of **1,2,…,n** to the vertices. Studied in algebraic statistics.
- **H** is a complete graph on **[n]** without loops: **n**-colorings of **G**. Obstructions of graph colorings.
- **H** is an edge with one loop: The independent sets of **G**. Defined by the stable set polytope. Hibi's ASL rings is a special case.



Examples of some theorems

- Maps between graphs induce maps between ideals (and quotients). Reflects on the Markov width.
- There is a quadratic square-free Gröbner base of $\mathbf{I}_{\mathbf{G}\to\mathbf{H}}$ if
  - the graph **G** is a forest, or
  - the graph **H** is an edge with one loop, and **G** is (almost) bipartite.
- For **H** an edge with one loop, any Markov width of $\mathbf{I}_{\mathbf{G}\to\mathbf{H}}$ is possible.



Read more about it here

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