Hierarchical Models, Markov Bases and Stanley-Reisner Ideals.

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(joint work with Sonja Petrović)

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  We call M the *design matrix* of the model

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Theorem (Diaconis-Sturmfels 1998)

Any generating set of the toric ideal I_M corresponds to a Markov basis of the model.

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- In terms of:
  - dimension
  - f-vector/h-vector
  - simplicial (co)homology

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  - *Decomposable* models have quadratic Markov bases. [Dobra 2000]
  - K_4-minor free graphs have Markov bases that live in degree at most 4 [Král'-Norine-Pangrác 2008].

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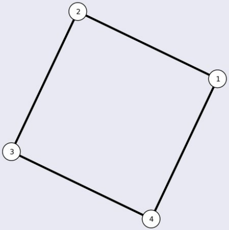
Example



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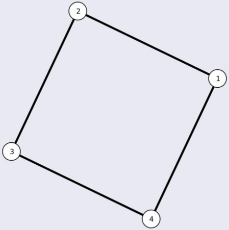

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- Knowing I_Δ is equivalent to knowing Δ.

Theorem (Petrovic-S.)

If $d = \operatorname{init} I_\Delta$ *then* $\operatorname{init} I_M = 2^{d-1}$

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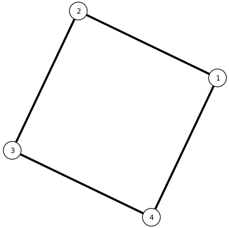

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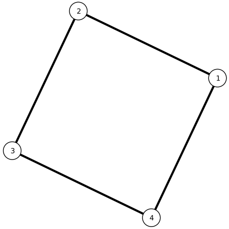

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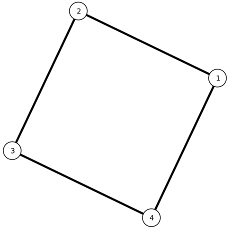

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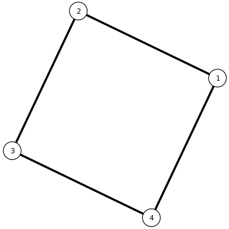

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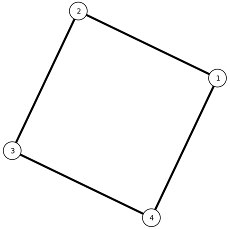

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Where in I_Δ are these degree 4 generators hiding?

- Let's look at the *Betti diagram* of Δ

	0	1	2
total:	1	2	1
0:	1	.	.
1:	.	2	.
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Vertex-decomposable includes all connected graphs and *matroids*.

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  Yes! I_M has a degree $2^2 = 4$ minimal generator

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- Actually computing the Markov basis takes a *LONG* time.

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- But the Markov basis also has elements with degree 6, 8, 10, 12.

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- When does the converse hold? (not always)
- For graphs, the converse can only hold if the last entry in the diagram is small. Does this generalize?