Bayes regularization and the geometry of discrete hierarchical loglinear models

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The problem

- We want to fit a hierarchical loglinear model to some discrete data given under the form of a contingency table.
- We put the Diaconis-Ylvisaker conjugate prior on the loglinear parameters of the multinomial distribution for the cell counts of the contingency table.
- We study the behaviour of the Bayes factor as the hyperparameter α of the conjugate prior tends to 0
- We are led to study the convex hull C of the support of the multinomial distribution.
- The faces of C are the most important objects in this study.

The data in a contingency table

- N objects are classified according to $|V|$criteria.
- We observe the value of $X = (X_\gamma | \ \gamma \in V)$ which takes its values (or levels) in the finite set I_γ.
- The data is gathered in a $|V|$-dimensional contingency table with

$$|I| = \times_{\gamma \in V} |I_\gamma| \ \text{ cells } \ i.$$

- The cell counts $(n) = (n(i), i \in \mathcal{I})$ follow a multinomial $\mathcal{M}(N, p(i), i \in \mathcal{I})$distribution.
- We denote $i_E = (i_\gamma, \gamma \in E)$ and $n(i_E)$ respectively the marginal-E cell and cell count.

The saturated loglinear model

We change the parametrization from $p(i), i \in \mathcal{I}$ to $\theta_i^E = \sum_{F \subset E} (-1)^{|E \setminus F|} \log p(i_F, 0_{F^c})$ so that

$$\log p(i) = \sum_{E \subseteq V} \theta_i^E,$$

where all possible interactions are taken into account.

We impose the baseline constraints: consider a special cell $i^* = 0 = (0, 0, \ldots, 0)$. Then, we impose

$$\theta_i^E = 0 \text{ if, for at least one } \gamma \in E,\ i_\gamma = 0.$$

For each $i \in I$, we define $S(i) = \{\gamma \in V :\ i_\gamma \neq 0\}$.
The loglinear model becomes

$$\log p(i) = \sum_{E \subseteq S(i)} \theta_i^E = \theta_i^\emptyset + \sum_{E \subseteq S(i), E \neq \emptyset} \theta_i^E.$$

The hierarchical loglinear model

- Let $\mathcal{A} = \{A_1, \dots, A_k\}$ be a class of subsets of V such that $\forall i, j,\ A_i \not\subseteq A_j$, and let

$$\mathcal{D} = \{E \neq \emptyset,\ E \subseteq A_i,\ \text{for some } i = 1, \dots, k\}$$

The hierarchical loglinear model with generating class $\mathcal{A}$ is

$$\log p(i) = \theta_i^{\emptyset} + \sum_{F \subseteq S(i), F \in \mathcal{D}} \theta_i^F \ .$$

- Let us introduce some convenient notation:

$$\begin{aligned} J &= \{j \in I, j \neq 0 = (0, \dots, 0), S(j) \in \mathcal{D}\} \\ \text{for } i \in I,\ j \lhd i \quad &\text{if} \quad S(j) \subseteq S(i),\ \text{and } j_{S(j)} = i_{S(j)} \\ J_i &= \{j \in J,\ j \lhd i\} \end{aligned}$$

Example

Consider the hierarchical model with

$$V = \{a, b, c\},\ \mathcal{A} = \{\{a, b\}, \{b, c\}\},\ I_a = \{0, 1, 2\} = I_b,\ \ I_c = \{0, 1\},$$

and $i = (0, 2, 1)$. We have

$$\begin{aligned}
\mathcal{D} &= \{a, b, c, ab, bc\} \\
J &= \{(1, 0, 0), (2, 0, 0), (0, 1, 0), (0, 2, 0), (0, 0, 1), (1, 1, 0), (1, 2, 0), \\
&\qquad (2, 1, 0), (2, 2, 0), (0, 1, 1), (0, 2, 1)\} \\
J_i &= \{(0, 2, 0), (0, 0, 1), (0, 2, 1)\} \\
\log p(0, 2, 1) &= \theta_i^{\emptyset} + \sum_{E \subseteq S(i), E \neq \emptyset} = \theta^{\emptyset}_{(0,2,1)} + \theta^{b}_{(0,2,1)} + \theta^{c}_{(0,2,1)} + \theta^{b,c}_{(0,2,1)} \\
&= \theta_{(0,0,0)} + \theta_{(0,2,0)} + \theta_{(0,0,1)} + \theta_{(0,2,1)} \\
&= \theta_0 + \sum_{j \in J_i} \theta_j
\end{aligned}$$

The multinomial hierarchical model

Since $J = \cup_{i \in \mathcal{I}} J_i$, the loglinear parameter is

$$\theta_J = (\theta_j, \;\; j \in J).$$

The hierarchical model is characterized by J. For $i \neq 0$, the loglinear model can then be written

$$\log p(i) = \theta_0 + \sum_{j \in J_i} \theta_j$$

with $\log p(0) = \theta_0$. Therefore

$$p(0) = e^{\theta_0} = (1 + \sum_{i \in I \setminus \{0\}} \exp \sum_{j \in J_i} \theta_j)^{-1} = L(\theta)^{-1}$$

and

$$\prod_{i \in I} p(i)^{n(i)} = \frac{1}{L(\theta)^N} \exp\{\sum_{j \in J} n(j_{S(j)} \theta_j\} = \exp\{\sum_{j \in J} n(j_{S(j)})\theta_j + N\theta_0\}.$$

The model as an exponential family

Make the change of variable

$$(n) = (n(i), i \in I \setminus \{0\}) \mapsto t = (t(i_E) = n(i_E), E \subseteq V \setminus \{\emptyset\}, i \in I \setminus \{0\}).$$

Then $\prod_{i \in I} p(i)^{n(i)}$ becomes

$$\begin{aligned} f(t_J|\theta_J) &= \exp\left\{ \sum_{j \in J} n(j_{S(j)})\theta_j - N \log(1 + \sum_{i \in I \setminus \{0\}} \exp \sum_{j \in J_i} \theta_j) \right\} \\ &= \frac{\exp \langle \theta_J, t_J \rangle}{L(\theta_J)^N} \text{ with } \theta_J = (\theta_j, j \in J), \;\; t_J = (n(j_{S(j)}, j \in J \end{aligned}$$

and $L(\theta_J) = (1 + \sum_{i \in I \setminus \{0\}} \exp \sum_{j \in J_i} \theta_j)$.
It is an NEF of dimension $|J|$, generated by the following measure.

The generating vectors

The set of functions from J to R is denoted by R^J and we write any function $h \in R^J$ as $h = (h(j), j \in J)$, which we can think of as a $|J|$ dimensional vector in $R^{|J|}$. Let $(e_j, j \in J)$ be the canonical basis of R^J and let

$$f_i = \sum_{j \in J, j \lhd i} e_j, \quad i \in I.$$

D	0	a	b	c	ab	ac	bc	abc
a	0	1	0	0	1	1	0	1
b	0	0	1	0	1	0	1	1
c	0	0	0	1	0	1	1	1
ab	0	0	0	0	1	0	0	1
bc	0	0	0	0	0	0	1	1

The measure

We note that in our example R^I is of dimension 8 while R^J is of dimension 5 and the $(f_j, j \in J)$ are, of course, 5-dimensional vectors. Consider now the counting measure in R^J

$$\mu_J = \delta_0 + \sum_{i \in \mathcal{I}} \delta_{f_i}.$$

For $\theta \in R^J$, the Laplace transform of μ_J is

$$\int_{R^J} e^{\langle \theta, x \rangle} \mu_J(dx) = 1 + \sum_{i \in \mathcal{I} \setminus \{0\}} e^{\langle \theta, f_i \rangle} = 1 + \sum_{i \in \mathcal{I} \setminus \{0\}} e^{\sum_{j \lhd i} \theta_j} = L(\theta).$$

Therefore the multinomial $f(t_J | \theta_J) = \frac{\exp\langle \theta_J, t_J \rangle}{L(\theta_J)^N}$ is the NEF generated by μ_J^{*N}.

C_J: The convex hull of the support of μ_J

Since $\mu_J = \delta_0 + \sum_{i \in \mathcal{I}} \delta_{f_i}$,

C_J is the open convex hull of $0 \in R^J$ and $f_j, j \in J$.

It is important to identify this convex hull since Diaconis and Ylvisaker (1974) have proven that the conjugate prior to an NEF, defined by

$$\pi(\theta_J | m_J, \alpha) = \frac{1}{I(m_J, \alpha)} e^{\{\alpha \langle \theta_J, m_J \rangle - \alpha \log L(\theta_J)\}}$$

is proper when the hyperparameters $m_J \in R^J$ and $\alpha \in R$ are such that

$$\alpha > 0 \text{ and } m_J \in C_J.$$

The DY conjugate prior

Clearly, we can write the multinomial density as $f(t_J|\theta_J) = f(t_J|\theta_J, J)$ where J represents the model. Assuming we put a uniform discrete distribution on the set of models,the joint distribution of J, t_J, θ_J is

$$f(J, t_J, \theta_J) \propto \frac{1}{I(m_J, \alpha)} e^{\{\langle \theta_J, t_J + \alpha m_J \rangle - (\alpha + N) \log L(\theta_J)\}}$$

and therefore the posterior density of J given t_J is

$$h(J|t_J) \propto \frac{I(\frac{t_J + \alpha m_J}{\alpha + N}, \alpha + N)}{I(m_J, \alpha)}.$$

The Bayes factor between two models

Consider two hierarchical models defined by J_1 and J_2. To simplify notation, we will write

$$h(J_k|t_{J_k}) \propto \frac{I(\frac{t_k+\alpha m_k}{\alpha+N}, \alpha + N)}{I(m_k, \alpha)}, \ k = 1, 2$$

so that the Bayes factor is

$$\frac{I(m_2, \alpha)}{I(m_1, \alpha)} \times \frac{I(\frac{t_1+\alpha m_1}{\alpha+N}, \alpha + N)}{I(\frac{t_2+\alpha m_2}{\alpha+N}, \alpha + N)}.$$

We will consider two cases depending on whether $\frac{t_k}{N} \in C_k, \ k = 1, 2$ or not.

The Bayes factor between two models

When $\alpha \to 0$, if $\frac{t_k}{N} \in C_k,\ k = 1, 2$, then

$$\frac{I(\frac{t_1+\alpha m_1}{\alpha+N}, \alpha + N)}{I(\frac{t_2+\alpha m_2}{\alpha+N}, \alpha + N)} \to \frac{I(\frac{t_1}{N}, N)}{I(\frac{t_2}{N}, N)}$$

which is finite. Therefore we only need to worry about $\lim \frac{I(m_2,\alpha)}{I(m_1,\alpha)}$ when $\alpha \to 0$.

When $\alpha \to 0$, if $\frac{t_k}{N} \in \bar{C}_k \setminus C_k,\ k = 1, 2$, then, we have to worry about both limits.

Limiting behaviour of $I(m,\alpha)$

Definitions. Assume C is an open nonempty convex set in R^n.

- The support function of C is $h_C(\theta) = \sup\{\langle\theta, x\rangle \ : \ x \in C\}$
- The characteristic function of C: $J_C(m) = \int_{R^n} e^{\langle\theta,m\rangle - h_C(\theta)} d\theta$

Examples of $J_C(m)$

- $C = (0,1)$. Then $h_C(\theta) = \theta$ if $\theta > 0$ and $h_C(\theta) = 0$ if $\theta \leq 0$. Therefore $h_C(\theta) = max(0,\theta)$ and

$$J_C(m) = \int_{-\infty}^{0} e^{\theta m} d\theta + \int_{0}^{+\infty} e^{\theta m - \theta} d\theta = \frac{1}{m(1-m)}.$$

Limiting behaviour of $I(m, \alpha)$

Examples of $J_C(m)$

- C is the simplex spanned by the origin and the canonical basis $\{e_1, \ldots, e_n\}$ in R^n and $m = \sum_{i=1}^n m_i e_i \in C$. Then

$$J_C(m) = \frac{n!\text{Vol}(C)}{\prod_{j=0}^n m_i} = \frac{1}{\prod_{j=0}^n m_i (1 - \sum_{j=0}^n m_i)}.$$

- $J = \{(1,0,0),(0,1,0),(0,0,1),(1,1,0),(0,1,1)\}$ with C spanned by $f_j, j \in J$ and $m = \sum_{j \in J} m_j f_j$. Then

$$\begin{aligned} J_C(m) &= \frac{m_{(0,1,0)}(1 - m_{(0,1,0)})}{D_{ab} D_{bc}} \\ D_{ab} &= m_{(1,1,0)}(m_{(1,0,0)} - m_{(1,1,0)})(m_{(0,1,0)} - m_{(1,1,0)})(1 - m_{(1,0,0)} - m_{(0,1,0)} + m_{(1,1,0)}) \\ D_{bc} &= m_{(0,1,1)}(m_{(0,0,1)} - m_{(0,1,1)})(m_{(0,1,0)} - m_{(0,1,1)})(1 - m_{(0,0,1)} - m_{(0,1,0)} + m_{(0,1,1)}) \end{aligned}$$

Limiting behaviour of $I(m, \alpha)$

Theorem

Let μ be a measure on R^n,$n = |J|,$ such that C the interior of the convex hull of the support of μ is nonempty and bounded. Let $m \in C$ and for $\alpha > 0$, let

$$I(m, \alpha) = \int_{R^n} \frac{e^{\alpha\langle\theta,m\rangle}}{L(\theta)^\alpha} d\theta.$$

Then

$$\lim_{\alpha\to 0} \alpha^n I(m, \alpha) = J_C(m).$$

Furthermore $J_C(m)$ is finite if $m \in C$.

Limit of the Bayes factor

Let models J_1 and J_2 be such that $|J_1| > |J_2|$ and the marginal counts $\frac{t_i}{N}$ are both in C_i. Then the Bayes factor

$$\frac{I(m_2,\alpha)}{I(m_1,\alpha)}\frac{I(\frac{t_1+\alpha m_1}{\alpha+N},\alpha+N)}{I(\frac{t_2+\alpha m_2}{\alpha+N},\alpha+N)} \sim \alpha^{|J_1|-|J_2|}\frac{I(\frac{t_1}{N},N)}{I(\frac{t_2}{N},N)}$$

Therefore the Bayes factor tends towards 0, which indicates that the model J_2 is preferable to model J_1.

We proved the heuristically known fact that taking α small favours the sparser model.

We can say that α close to "0 " regularizes the model.

Some comments

If $\frac{t_i}{N}$ are both in $C_i, i = 1, 2$ and $|J_1| \neq |J_2|$, we need not compute $J_C(m)$.

If $\frac{t_i}{N}$ are both in $C_i, i = 1, 2$ and $|J_1| = |J_2|$, then we might want to compute $J_C(m_i) i = 1, 2$. In this case, we have a few theoretical results. We define the polar convex set C_0 of C

$$C^0 = \{\theta \in R^n \; ; \; \langle \theta, x \rangle \leq 1 \;\; \forall x \in C\}$$

then

- $\frac{J_C(m)}{n!} = \text{Vol}(C - m)^0$
- If C in R^n is defined by its K $(n-1)$-dimensional faces $\{x \in R^n : \langle \theta_k, x \rangle = c_k\}$, then for $D(m) = \prod_{k=1}^{K} (\langle \theta_k, x \rangle - c_k)$,

$$D(m) J_C(m) = N(m)$$

where degree of $N(m)$ is $< K$.

Limiting behaviour of $I(\frac{\alpha m+t}{\alpha+N}, \alpha + N)$

We now consider the case when $\frac{t}{N}$ belongs to the boundary of C. Then each face of $\bar{C}$ of dimension $|J| - 1$ is of the form

$$F_g = \{x \in \bar{C} : g(x) = 0\}$$

where g be an affine form on R^J.

<u>Theorem</u>

Suppose $\frac{t}{N} \in \bar{C} \setminus C$ belongs to exactly M faces of $\bar{C}$. Then

$$\lim_{\alpha \to 0} \alpha^{\min(M,|J|)} I(\frac{\alpha m + t}{\alpha + N}, \alpha + N)$$

exists and is positive.

The Bayes factor

Combining the study of the asymptotic behaviour of $I(m, \alpha)$ and $I(\frac{\alpha m + t}{\alpha + N}, \alpha + N)$, we obtain that

when $\alpha \to 0$, the Bayes factor behaves as follows

$$\frac{I(m_2, \alpha)}{I(m_1, \alpha)} \frac{I(\frac{t_1 + \alpha m_1}{\alpha + N}, \alpha + N)}{I(\frac{t_2 + \alpha m_2}{\alpha + N}, \alpha + N)}$$

$$\sim \quad C \alpha^{|J_1| - |J_2| - [\min(M_1, |J_1|) - \min(M_2, |J_2|)]} \frac{J_{C_1}(m_1)}{J_{C_2}(m_2)}$$

where C is a positive constant.

Some faces of C for graphical models

For graphical models, let $\mathcal{C}$ be the set of cliques of the graph G.

For each $D \in \mathcal{C}$ and each $j_0 \in J$ such that $S(j_0) \subset D$ define

$$\begin{aligned} g_{0,D} &= \sum_{j;S(j)\subset D} (-1)^{|S(j)|} e_j \\ g_{j_0,D}(m) &= \sum_{j;S(j)\subset D,\ j_0 \lhd j} (-1)^{|S(j)|-|S(j_0)|} e_j \end{aligned}$$

and the affine forms

$$\begin{aligned} g_{0,D}(t) &= 1 + \langle g_{0,D}, t \rangle \\ g_{j_0,D}(t) &= \langle g_{j_0,D}, t \rangle. \end{aligned}$$

Some faces of C

All subsets of the form

$$F(j,D) = H(j,D) \cap \overline{C}$$

with

$$H(j,D) = \{t \in \mathrm{R}^J \;;\; g_{j,D}(t) = 0\},\ D \in \mathcal{C},\ S(j) \subset D$$

are faces of C

Example $a - - - b - - - c$. The faces are

$$t_{ab} = 0,\ t_a - t_{ab} = 0,\ t_b - t_{ab} = 0,\ 1 - t_a - t_b + t_{ab} = 0$$

and

$$t_{bc} = 0,\ t_b - t_{bc} = 0,\ t_c - t_{bc} = 0,\ 1 - t_b - t_c + t_{bc} = 0.$$

The faces of C when G is decomposable

For decomposable models,

$$H(j, D) = \{m \in \mathsf{R}^J \ ; \ g_{j,D}(m) = 0\}, \ D \in \mathcal{C}, \ S(j) \subset D$$

are the only faces of C.

Example $a - - - b - - - c$. The faces are

$$t_{ab} = 0, j = (1,1,0) \ t_a - t_{ab} = 0, j = (1,0,0)$$
$$t_b - t_{ab} = 0, j = (0,1,0) \ 1 - t_a - t_b + t_{ab} = 0, S(j) = \emptyset$$
$$t_{bc} = 0, j = (0,1,1) \ t_b - t_{bc} = 0, j = (0,1,0)$$
$$t_c - t_{bc} = 0, j = (0,0,1) \ 1 - t_b - t_c + t_{bc} = 0 S(j) = \emptyset.$$

Some faces when G is a cycle

Theorem Let $G = (V, E)$ be a cycle of order n. Let (a, b) be an edge of the cycle. Then the hyperplanes

$$\langle s_{ab}, t\rangle = -t_a - t_b + 2t_{ab} + \sum_c t_c - \sum_{e \in E} t_e = \begin{cases} 0 \\ a_n \end{cases}$$

where $a_n = \frac{n-1}{2}$ if n is odd and $a_n = \frac{n-2}{2}$ when n is even, define faces of C.

Example of model search

We study the Czech Autoworkers 6-way table from Edwards and Havranek (1985).

This cross-classfication of 1841 men considers six potential risk factors for coronary trombosis:

- a, smoking;
- b, strenuous mental work;
- c, strenuous physical work;
- d, systolic blood pressure;
- e, ratio of beta and alpha lipoproteins;
- f, family anamnesis of coronary heart disease.

Edwards and Havranek (1985) use the LR test and Dellaportas and Forster (1999) use a Bayesian search with normal priors on the θ to analyse this data.

Czech Autoworkers example our method

We use a Bayesian search with

- MC^3
- our prior with $\alpha = 1, 2, 3, 32$ and then $\alpha = .05, .01$ and equal fictive counts for each cell
- The Laplace approximation to the marginal likelihood

Czech Autoworkers example

Search	$\alpha = 1$		$\alpha = 2$	
Dec.	$bc\|ace\|ade\|f$	0.250	$bc\|ace\|ade\|f$	0.261
	$bc\|ace\|de\|f$	0.104	$bc\|ace\|de\|f$	0.177
	$bc\|ad\|ace\|f$	0.102	$bc\|ace\|de\|bf$	0.096
	$ac\|bc\|be\|de\|f$	0.060	$bc\|ad\|ace\|f$	0.072
	$bc\|ace\|de\|bf$	0.051	$bc\|ace\|de\|bf$	0.065
	$bc\|ace\|de\|f$	med	$bc\|ad\|ace\|de\|f$	med
Graph.	$ac\|bc\|be\|ade\|f$	0.301	$ac\|bc\|be\|ade\|f$	0.341
	$ac\|bc\|ae\|be\|de\|f$	0.203	$ac\|bc\|be\|ade\|bf$	0.141
	$ac\|bc\|be\|ade\|bf$	0.087	$ac\|bc\|ae\|be\|de\|f$	0.116
	$ac\|bc\|ad\|ae\|be\|f$	0.083	$ac\|bc\|be\|ade\|ef$	0.059
	$ac\|bc\|ae\|be\|de\|bf$	0.059		
	$ac\|bc\|ad\|ae\|be\|de\|f$	med	$ac\|bc\|be\|ade\|f$	med
Hierar.	$ac\|bc\|ad\|ae\|ce\|de\|f$	0.241	$ac\|bc\|ad\|ae\|ce\|de\|f$	0.175
	$ac\|bc\|ad\|ae\|be\|de\|f$	0.151	$ac\|bc\|ad\|ae\|be\|de\|f$	0.110
	$ac\|bc\|ad\|ae\|be\|ce\|de\|f$	0.076	$ac\|bc\|ad\|ae\|be\|ce\|de\|f$	0.078
	$ac\|bc\|ad\|ae\|ce\|de\|bf$	0.070	$ac\|bc\|ad\|ae\|ce\|de\|bf$	0.072
	$ac\|bc\|ad\|ae\|ce\|de\|f$	med	$ac\|bc\|ad\|ae\|be\|ce\|de\|f$	med

Results for α close to 0

Search	$\alpha = .5$		$\alpha = .01$	
Hierar.	$ac\|bc\|ad\|ae\|ce\|de\|f$	0.3079	$ac\|bc\|ad\|ae\|ce\|de\|f$	0.2524
	$ac\|bc\|ad\|ae\|be\|de\|f$	0.1926	$ac\|bc\|ad\|ae\|be\|de\|f$	0.1577
	$ac\|bc\|ad\|ae\|be\|ce\|de\|f$	0.0686	$ac\|bc\|ae\|ce\|de\|f$	0.1366
	$ac\|bc\|ad\|ae\|ce\|de\|be$	0.0631	$ac\|bc\|d\|ae\|ce\|f$	0.1168
	$ac\|bc\|ad\|ae\|ce\|de\|f$	med	$ac\|bc\|ae\|de\|f$	0.0854
			$ac\|bc\|c\|ae\|be\|f$	0.0730
			$ac\|bc\|ad\|ae\|ce\|f$	0.0558

Recall that for $\alpha = 1, 2$, the most probable model was $ac|bc|ad|ae|ce|de|f$ with respective probablities 0.241 and 0.175.

As $\alpha \mapsto 0$, the models become sparser but are consistent with those corresponding to larger values of α.

Another example

32	3	86	2	56	35	7	0
130	12	59	5	142	91	5	0

Marginal a, b, d, h table from the Rochdale data in whittaker1990. The cells counts are written in lexicographical order with h varying fastest and a varying slowest.

The three models considered

We will consider three models J_0, J_1 and J_2 such that

(a) J_0 is decomposable with cliques $\{a, d\}, \{d, b\}, \{b, h\}$ so that $\mathcal{D}$ as defined in Section 2 is

$$\mathcal{D}_0 = \{a, b, d, h, (ad), (db), (bh)\}, \ |J_0| = 7, \ M_0 = 0.$$

(b) J_1 is a hierarchical model with generating set $\{(ad), (bd), (bh), (dh)\}$. This is not a graphical model and

$$\mathcal{D}_1 = \{a, b, d, h, (ad), (db), (bh), (dh)\}, \ |J_1| = 8 \ M_1 = 0.$$

(c) J_2 is decomposable with cliques $\{b, d, h\}, \{a\}$,and

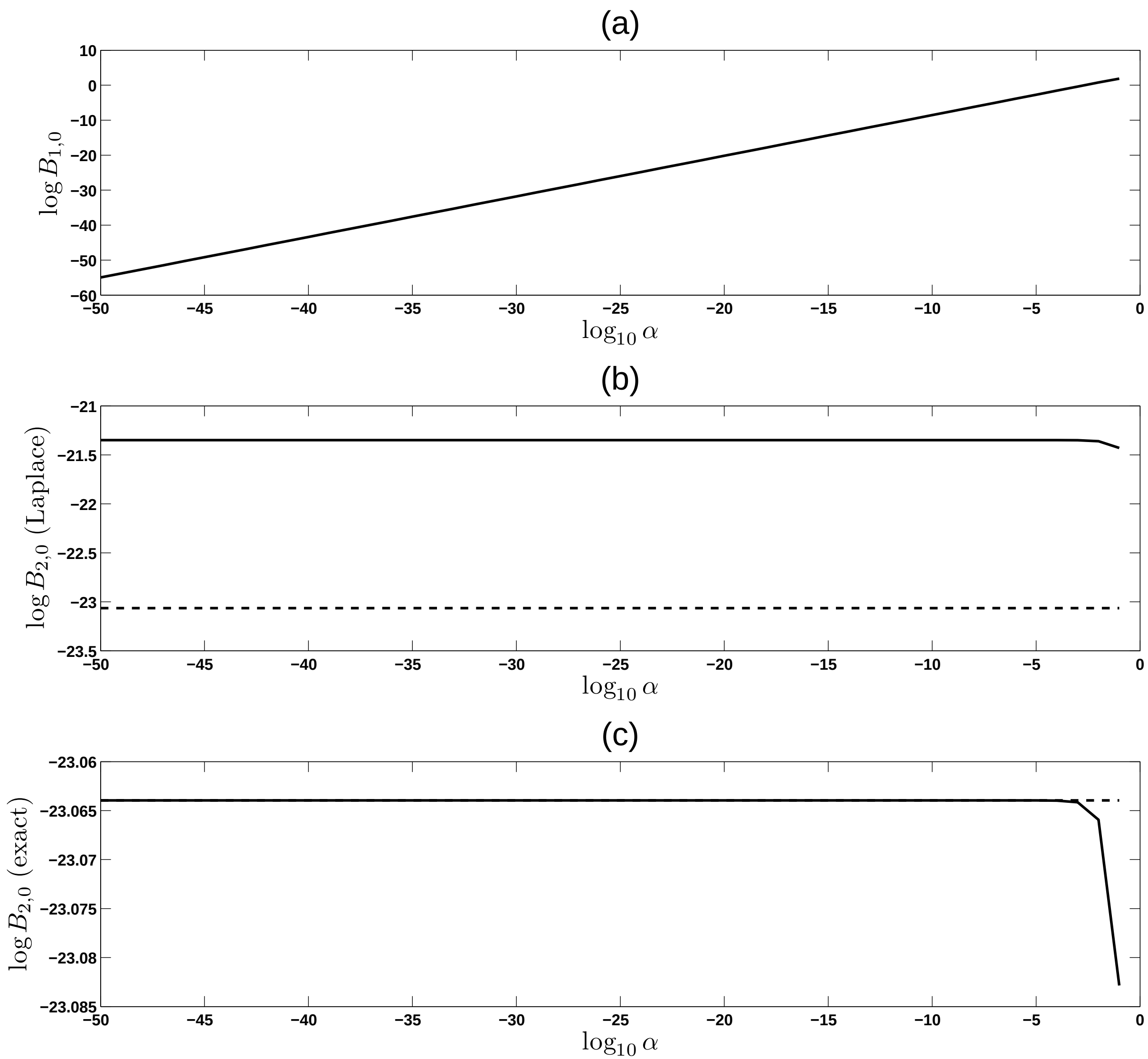
$$\mathcal{D}_2 = \{a, b, d, h, (ad), (db), (bh), (dh), (bdh)\}, \ |J_2| = 8, \ M_2 = 1.$$

Asymptotics of $B_{1,0}$ and $B_{2,0}$

We have

$$\begin{aligned} B_{1,0} &\sim \alpha^{|J_0|-|J_1|-[\min(M_0,|J_0|)-\min(M_1,|J_1|)]} \frac{J_{C_1}(m_1)}{J_{C_0}(m_0)} \\ &= C_{1,0}\alpha^{(7-8-(0-0)} = C\alpha^{-1} \\ B_{2,0} &\sim \alpha^{|J_0|-|J_2|-[\min(M_0,|J_0|)-\min(M_2,|J_2|)]} \frac{J_{C_2}(m_2)}{J_{C_0}(m_0)} \\ &= C_{2,0}\alpha^{(7-8-(0-1)} = C_{2,0}\alpha^{0} = C_{2,0} \end{aligned}$$

The graphs


(a)
10
0
−10
−20
−30
−40
−50
−60
$\log B_{1,0}$
−50
−45
−40
−35
−30
−25
−20
−15
−10
−5
0
$\log_{10}\alpha$
(b)
−21
−21.5
−22
−22.5
−23
−23.5
$\log B_{2,0}$ (Laplace)
$\log_{10}\alpha$
(c)
−23.06
−23.065
−23.07
−23.075
−23.08
−23.085
$\log B_{2,0}$ (exact)
$\log_{10}\alpha$