Binomial Edge Ideals and Conditional Independence

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Binomial Edge Ideals

Undirected Graphs

- G an undirected simple graph with vertex set $[n]$.
- $S = \Bbbk[x_1, \ldots, x_n, y_1, \ldots, y_n]$ polynomial ring in $2n$-variables.
- For any edge $\{i, j\}$ with $i < j$:

$$f_{ij} := x_i y_j - x_j y_i.$$

Definition

The binomial edge ideal associated to G is

$$J_G := \langle f_{ij} : \{i, j\} \in E(G), i < j \rangle \subseteq S$$

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Remarks

- All generators are (2×2)-minors of

$$\begin{pmatrix} x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \end{pmatrix}$$

- If G is a path: Ideal of adjacent minors.
- Disregard isolated vertices

Algebra of Binomial Edge Ideals

For the CI-Application we would want the following anwers:

Questions

- Gröbner basis
- Primary(prime?) Decomposition
- Relation to graph properties
- Cohen-Macaulayness of S/J_G.

Term order

In the following we fix lexicographic term order induced by

$$x_1 > x_2 > \dots x_n > y_1 > y_2 > \dots y_n$$

Gröbner basis

Let $i < j$ be two vertices of G

Definition

A path $i = i_0, i_1, \ldots, i_r = j$ from i to j is called *admissible*, if

(i) $i_k \neq i_\ell$ for $k \neq \ell$;

(ii) for each $k = 1, \ldots, r-1$ one has either $i_k < i$ or $i_k > j$;

(iii) for any proper subset $\{j_1, \ldots, j_s\}$ of $\{i_1, \ldots, i_{r-1}\}$, the sequence $i, j_1, \ldots, j_s, j$ is not a path.

We assign a monomial to $\pi : i = i_0, i_1, \ldots, i_r = j$:

$$u_\pi = (\prod_{i_k > j} x_{i_k})(\prod_{i_\ell < i} y_{i_\ell}).$$

Gröbner basis II

Theorem

The set of binomials

$$\mathcal{G} = \bigcup_{i<j} \{\, u_\pi f_{ij} \;:\; \pi \text{ is an admissible path from } i \text{ to } j \,\}$$

is a reduced Gröbner basis of J_G*.*

Proof: Work out all S-pairs!

Corollary

- J_G is a radical binomial ideal for any G.

Quadratic Gröbner basis

Quadratic Gröbner bases

The generators f_{ij} form a quadratic Gröbner basis of J_G if and only if for any pair $\{i,j\}, \{k,l\} \in E(G)$ ($i < j$, $k < l$).

- $i = k \Rightarrow \{j,l\} \in E(G)$,
- $j = l \Rightarrow \{i,k\} \in E(G)$.

"Shortcutting" all admissible paths.

Remarks

- The condition depends on the numbering of the vertices (and the isomorphism type) of G.
- It's a condition on the associated DAG.

Closed Graphs

To each simple undirected graph we associate the digraph with edges (i, j) such that $i < j$. By construction it is a DAG.

Closed Graphs

A simple undirected graph G is called closed if for any edges $\{i, j\}, \{k, l\} \in E(G)$ ($i < j$, $k < l$):

- $i = k \Rightarrow \{j, l\} \in E(G)$,
- $j = l \Rightarrow \{i, k\} \in E(G)$.

Proposition

G is closed iff in the associated DAG every shortest path is directed.

Remarks

- Closed $\Rightarrow$ chordal and claw free.
- Each G has a "closure".
- Closed + 1 additional condition $\Rightarrow$ CM of S/J_G.

Minimal Primes

Minimal primes are characterized by the monomials they contain:

Feasible supports

- Consider subsets $W \subseteq V(G)$.
- Denote C_W the connected components of induced graph on $V \setminus W$.
- W is called feasible if $\left|C_{W \setminus i}\right| < |C_W|$ for any $i \in W$.
- Feasible sets W describe monomials in prime components.

Theorem

Let $M_W := \langle x_i, y_i : i \in W \rangle$. The minimal primes of J_G are

$$(J_G + M_W) : (\prod_{i \notin W} x_i y_i)^\infty$$

where W runs over the feasible subsets of vertices.

Conditional Independence in Random Vectors

Basics of Discrete Random Vectors

- Vector $X = (X_0, \ldots, X_N)$ of $N+1$ discrete random variables.
- X takes values in $\mathcal{X} = \prod_{i=0}^{N} [d_i]$, $d_0 = 2$.
- For $K \subseteq [N]$, X_K in $\mathcal{X}_K = \prod_{i \in K} [d_i]$.
- A joint distribution of X: non-negative real valued vector $p \in \mathbb{R}^{\mathcal{X}}$, such that $\sum_{x \in \mathcal{X}} p(x) = 1$.
- Sometimes denote $p(x) = p_{x_0 x_1 \ldots x_n}$.
- $\mathbb{C}[p_x : x \in \mathcal{X}]$ has two groups of variables: $x_0 = 1$, and $x_0 = 2$.

Conditional Independence

CI-Statements

- Consider CI-statements of the form

$$X_0 \perp\!\!\!\perp X_K \,\big|\, X_{\overline{K}}\,, \quad \emptyset \neq K, \overline{K} := [N] \setminus K. \tag{1}$$

- A distribution p satisfies (1) iff

$$p(1, x_K, x_{\overline{K}})p(2, x'_K, x_{\overline{K}}) = p(1, x'_K, x_{\overline{K}})p(2, x_K, x_{\overline{K}}),$$

for all $x_K, x'_K \in \mathcal{X}_K, x_{\overline{K}} \in \mathcal{X}_{\overline{K}}$

Output Robustness Models

- "Output" X_0 independent of some inputs, given the remaining ones.
- CI-Ideals using only these statements are binomial edge ideals!

CI and Binomial Edge Ideals

Robustness Ideals are Binomial Edge Ideals

- 2 Groups of variables p_x with $x_0 = 1, 2$.
- All CI-Equations are (2×2)-minors of

$$\begin{pmatrix} p_{11\ldots1} & p_{11\ldots2} & \cdots & p_{1d_1\ldots d_n} \\ p_{21\ldots1} & p_{21\ldots2} & \cdots & p_{2d_1\ldots d_n} \end{pmatrix}.$$

- The graph G consists of
  - Vertex set : $\mathcal{X}_{[N]} = \prod_{i=1}^{N} [d_i]$.
  - Edges set for statement $X_0 \perp\!\!\!\perp X_K \,|X_{\overline{K}}$:

$$x \sim x' \quad \Leftrightarrow \quad x \text{ and } x' \text{ differ in } K \text{ only.}$$

- We will consider the vertex set as ordered!

Conditional Independence Models

Independence Model

Assume that each $i \in [N]$ is contained in at least one of the K_j.

- Empty set (feasible!) corresponds to the toric ideal $J_G : (\prod_{i=1}^n x_i y_i)^\infty$. ("Intersection axiom"): $X_0 \perp\!\!\!\perp X_{[N]}$ (All (2×2)-minors)
- Parameterizations of the parts: Let $W \neq \emptyset$ be feasible, λ any probability distribution on connected components C_W and $p : \mathcal{Y} \times \mathcal{C} \to [0,1]$ a conditional distribution of y given a component from C_W, then

$$p_{x_0 x_1 \dots x_n} = \begin{cases} \lambda(B) p(x_0|B) & \forall B \in C_W, \forall (x_1, \dots, x_n) \in B, x_0 = 1,2, \\ 0 & (x_1, \dots, x_n) \notin \bigcup_{B \in C_W} B \end{cases}$$

  is a distribution from the component corresponding to C, and every distribution arises in this way.

Example: Three Binary Inputs

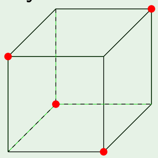
3 Binary Inputs

- (X_0, X_1, X_2, X_3) binary.
- $X_0 \perp\!\!\!\perp X_i \,\big|\, X_{[N]\setminus\{i\}}$, $i = 1, 2, 3.$
- This models robustness against single input knockouts.

Feasible supports

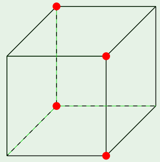
The empty set is feasible $\Rightarrow$ full support

Totally disconnected



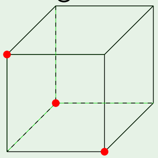
$(2\times)$

Cutting plane



$(6\times)$

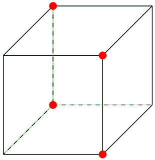
Cutting a vertex



$(8\times)$

Example (continued)

Cutting plane



Features

- supported on $p(x_0111)$, $p(x_0112)$, $p(x_0221)$, $p(x_0222)$,
- "surviving relations" on edges:

$$p(1111)p(2112) = p(2111)p(1112)$$
$$p(1221)p(2222) = p(2221)p(1222)$$

- A distribution in this component is of the form

$$p(x_0x_1x_2x_3) = \lambda(B)p(x_0|B)$$

for $B \in \{\{x_{12} = 11\}, \{x_{12} = 22\}\}$.

Conclusion

Binomial Edge Ideals

- Nice class of radical CI-Ideals
- Generalize:
  - Ideals of adjacent minors of a $(2 \times n)$-matrix.
  - CI-Ideals studied by A. Fink.

Non-binary X_0

- Minimal primes done
- Radicality almost done.
- Gröbner basis: good conjecture.

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Thank you for your attention.